7/12/2020

MATRICES

Matrix: A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns is called a matrix.

Types of Matrices:

(i.) Rectangular matrix
(ii.) Square matrix
(iii.) Row matrix
(iv.) Column matrix
(v.) Null matrix
(vi.) Unit matrix
(vii.) Principal diagonal of square matrix
(viii.) Trace of matrix
(ix.) Diagonal matrix
(x.) Scalar matrix
(xi.) Triangular matrix
(a.) Upper triangular matrix
(b.) Lower triangular matrix

Nilpotent Matrix: A square matrix A is called as a nilpotent matrix if $\boxed{A^n = O}$; $n \in N$.

Idempotent Matrix: A square matrix A is called as idempotent matrix if $\boxed{A^2 = A}$.

Involutory Matrix: A square matrix A is called an involutory matrix if $\boxed{A^2 = I}$

Transpose of Matrix:

(i.) $A = B \Leftrightarrow A^T = B^T$

(ii.) $I^T = I$

(iii.) $(A^2)^T = (A^T)^2$

Properties:

(i.) $(A^T)^T = A$

(ii.) $(KA)^T = K(A)^T$

(iii.) $(A+B)^T = A^T + B^T$

(iv.) $(AB)^T = B^T A^T$

Symmetric Matrix: If A is square matrix and $\boxed{A^T = A}$ then A is said be symmetric matrix.

Skew-Symmetric Matrix: If A is a square matrix and $\boxed{A^T = -A}$ then A is said to be skew-symmetric matrix.

* $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$ i.e., every square matrix can be expressed as sum of symmetric and skew-symmetric matrix. — ①

Orthogonal Matrix: A is a square matrix and $\boxed{AA^T = A^TA = I}$ then A is said to be orthogonal matrix.

* Condition to posses inverse of matrix A:

$\boxed{\det A \neq 0}$ then $\boxed{A^{-1} = \frac{\operatorname{Adj} A}{\det A}}$

Proof: ① $A = \underbrace{\frac{1}{2}(A+A^T)}_{P} + \underbrace{\frac{1}{2}(A-A^T)}_{Q}$

Now: $P^T = \left(\frac{1}{2}(A+A^T)\right)^T = \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A^T+A)$

$\boxed{P^T = P}$

$\therefore$ P is symmetric matrix.

$Q^T = \left(\frac{1}{2}(A-A^T)\right)^T = \frac{1}{2}(A-A^T)^T = \frac{1}{2}(A^T-A) = -\frac{1}{2}(A-A^T)$

$\boxed{Q^T = -Q}$ $\therefore$ Q is skew-symmetric matrix

Hence proved.

*. If A is a matrix of order $n \times n$ then

(i.) $adj(KA) = K^{n-1}(adj A)$, $K \in R$

(ii) $|adj A| = |A|^{n-1}$

(iii.) $adj(adj A) = |A|^{n-2} A$; $|A| \neq 0$

(iv.) $|adj(adj A)| = |A|^{(n-1)^2}$.

8/12/2020

*. Inverse of non-singular symmetric matrix A is symmetric.

Proof:- A is non-singular matrix then $\boxed{|A| \neq 0}$

A^{-1} exist. Given $\boxed{A = A^T}$

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

$\boxed{(A^{-1})^T = A^{-1}}$ $\therefore$ A^{-1} is symmetric matrix.

*. If A is symmetric matrix then prove that $Adj A$ is symmetric matrix.

Proof:- A is symmetric matric then $\boxed{A^T = A}$

$$\boxed{A^{-1} = \frac{Adj A}{det A}}$$

$$Adj A = A^{-1}(det A)$$

$$(Adj A)^T = (A^{-1}(det A))^T = (A^{-1})^T det A = (A^T)^{-1} det A$$

$$(Adj A)^T = A^{-1}(det A) = Adj A$$

$$\boxed{(Adj A)^T = Adj A}$$

Hence proved.

★. If A and B are orthogonal matrix each of order n then prove that AB and BA are orthogonal matrix.

Proof: A and B are orthogonal matrix.

then

$$A^TA = AA^T = I \quad —(1)$$

$$BB^T = B^TB = I \quad —(2)$$

Now:- $(AB)^T = B^TA^T$ (∵ from ① & ②)

$$(AB)^T(AB) = B^TA^T(AB) = B^TB(AA^T) = B^TB(I)$$

$$(AB)^T(AB) = BB^T = I$$

∴ AB is orthogonal matrix. Similarly BA is orthogonal matrix.

★. If A is orthogonal matrix then prove that $|A| = \pm 1$.

Proof:- $AA^T = A^TA = I$

Consider :- $AA^T = I$

$$|A||A^T| = |I|$$

$$|A|^2 = 1$$

$$\boxed{|A| = \pm 1}$$

Hence proved.

Example:- Express matrix $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 1 & -1 \\ 5 & 4 & 0 \end{bmatrix}_{3\times3}$ in the form of sum of symmetric and skew-symmetric matrix.

Sol:- Given:- $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 1 & -1 \\ 5 & 4 & 0 \end{bmatrix}_{3\times3}$ then $A^T = \begin{bmatrix} 3 & 2 & 5 \\ -2 & 1 & 4 \\ 6 & -1 & 0 \end{bmatrix}_{3\times3}$

We know that:- $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = P+Q$

$$P = \frac{1}{2}(A+A^T) = \frac{1}{2}\begin{bmatrix} 6 & 0 & 11 \\ 0 & 2 & 3 \\ 11 & 3 & 0 \end{bmatrix}_{3\times3} = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 1 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}_{3\times3}$$

$$P^T = \begin{bmatrix} 3 & 0 & 11/2 \\ 0 & 1 & 3/2 \\ 11/2 & 3/2 & 0 \end{bmatrix}_{3\times3} = P$$

$\boxed{P = P^T}$ $\therefore$ P is Symmetric matrix.

$$Q = \frac{1}{2}(A-A^T) = \frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix}_{3\times3} = -\frac{1}{2}\begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$$

$$Q^T = -\frac{1}{2}\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -5 \\ -1 & 5 & 0 \end{bmatrix} = -Q$$

$\boxed{Q^T = -Q}$ $\therefore$ Q is skew symmetric matrix.

Example:- If $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}_{3\times3}$ is orthogonal or not?

Sol:-

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}_{3\times3} \text{ then } A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}_{3\times3}$$

Orthogonal matrix :- Condition $AA^T = A^TA = I$

Consider :- $A \cdot A^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}_{3\times3} \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}_{3\times3}$

$$A \cdot A^T = \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix}_{3\times3}$$

$$AA^T = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix}_{3\times3} \neq I$$

$\therefore$ A is non-orthogonal matrix.

* Conjugate matrix :- If A is matrix containing complex numbers then the conjugate matrix of A is denoted by $\bar{A}$.

Ex :- $A = \begin{bmatrix} 2 & 2+i \\ i-1 & 3 \end{bmatrix}_{2\times2} \Rightarrow \bar{A} = \begin{bmatrix} 2 & 2-i \\ -i-1 & 3 \end{bmatrix}_{2\times2}$

$\Rightarrow$ If $A = [a_{ij}]_{m\times n}$ then $\bar{A} = [b_{ij}]_{m\times n}$

$$\boxed{b_{ij} = \overline{a_{ij}}}$$

$\Rightarrow$ If $\bar{A}$ & $\bar{B}$ Conjugate matrix of A & B :-

(1.) $(\overline{\bar{A}}) = A$ (2.) $(\overline{A+B}) = \bar{A} + \bar{B}$ (3.) $(\overline{KA}) = K(\bar{A})$

* If A is square matrix, $\bar{A}$ is conjugate of A then transpose of $\bar{A} = (\bar{A})^T = A^\theta = (\overline{A^T})$

Then Properties:-

(1.) $(A^\theta)^\theta = A$ (3.) $(KA)^\theta = \bar{K}(A^\theta)$

(2.) $(A \pm B)^\theta = A^\theta \pm B^\theta$ (4.) $(AB)^\theta = B^\theta A^\theta$

Hermitian Matrix: A square matrix A said to be a hermitian matrix if $\boxed{A^\theta = A}$ then $\boxed{\bar{A} = A^T}$ or $\boxed{(\bar{A})^T = A}$

→ Principle diagonal elements are real.

Skew-Hermitian Matrix: A square matrix A said to be a skew-hermitian matrix if $\boxed{A^\theta = -A}$ then $\boxed{A^T = -\bar{A}}$ or $\boxed{(\bar{A})^T = -A}$

→ Principle diagonal element are either zero (or) purely imaginary.

Unitary matrix: A square matrix A is said to be to be unitary, if $\boxed{A^\theta A = AA^\theta = I}$.

Ex:- $A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix} \Rightarrow \bar{A} = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$

$(\bar{A})^T = \bar{A} = A^\theta$ $\boxed{\because i^2 = -1}$

$$AA^\theta = \begin{bmatrix} \frac{-i^2+3}{4} & \frac{i\sqrt{3} - \sqrt{3}i}{4} \\ \frac{-\sqrt{3}i + \sqrt{3}i}{4} & \frac{3 - i^2}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\boxed{AA^\theta = I = A^\theta A}$$

* $A = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$ i.e., every square matrix is expressed in the form of sum of hermitian matrix and skew hermitian matrix.

Proof:- Consider:- $P = \frac{1}{2}(A+A^\theta)$; $Q = \frac{1}{2}(A-A^\theta)$

$$P^\theta = \frac{1}{2}(A+A^\theta)^\theta = \frac{1}{2}(A^\theta + A) = P$$

$\boxed{P^\theta = P}$ $\therefore$ P is hermitian matrix.

$$Q^\theta = \frac{1}{2}(A-A^\theta)^\theta = \frac{1}{2}(A^\theta - A) = \frac{-1}{2}(A-A^\theta) = -Q$$

$\boxed{Q^\theta = -Q}$ $\therefore$ Q is skew-hermitian matrix.

* Let us consider, $A = R+S$

R = hermitian matrix ; S = skew-hermitian matrix

$A^\theta = R^\theta + S^\theta = (R+S)^\theta = R-S$

$\Rightarrow A^\theta = R-S$

$\Rightarrow A + A^\theta = R - S + A$

$\Rightarrow \frac{1}{2}(A+A^\theta) = \frac{1}{2}(R - \cancel{S} + R + \cancel{S})$

$\boxed{\frac{1}{2}(A+A^\theta) = R}$

$A^\theta = (R+S)^\theta$

$-A^\theta = \cancel{R^\theta + S^\theta}\ R^\theta + S^\theta$

$\frac{1}{2}(A-A^\theta) = \frac{1}{2}(R + S - R^\theta + S^\theta$

$\frac{1}{2}(A-A^\theta) = \frac{1}{2}(R+S-R$

$\boxed{\frac{1}{2}(A-A^\theta) = S}$

$\therefore \frac{1}{2}(A+A^\theta)$ is hermitian matrix.

$\frac{1}{2}(A-A^\theta)$ is skew-hermitian matrix.

9/12/2020

(1.) If A is a hermitian matrix iA is skew-hermiti[an]

Proof:- Given:- $\boxed{A^{\theta} = A}$ $\Rightarrow$ then $(\bar{A})^{T} = A^{\theta}$

R.T.P :- $(iA)^{\theta} = -(iA)$

Consider:-

$\Rightarrow (iA)^{\theta} = (iA)^{\theta} = (\overline{iA})^{T} = -i(\bar{A})^{T} = -iA^{\theta}$

$$\boxed{(iA)^{\theta} = -(iA)}$$

Hence proved.

(2.) If A is skew-Hermitian matrix then show that iA is hermitian matrix.

Proof. Given. $\boxed{A^{\theta} = -A}$

then Consider:- $(iA)^{\theta} = (\overline{iA})^{T} = -i(\bar{A})^{T} = -iA^{\theta} = -i(-A)$

$$\boxed{(iA)^{\theta} = (iA)}$$

$\therefore$ Hence proved.

(3.) If $A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}_{3\times 3}$ then express it as sum of hermitian and skew-hermitian matrix.

[So]l. Let, $\boxed{A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})}$

Since $\boxed{A^{\theta} = (\bar{A})^{T}}$

[L]et

$\frac{1}{2}(A + A^{\theta}) = P =$ herimitian matrix.

$\frac{1}{2}(A - A^{\theta}) = Q =$ skaw-hermitian matrix.

$$(\bar{A}) = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}_{3\times 3}$$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}_{3\times 3}$$

$$P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}\begin{bmatrix} 2 & 2-2i & 4-6i \\ 2i+2 & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}_{3\times 3}$$

$$\therefore \quad P = \begin{bmatrix} 1 & 1-i & 2-3i \\ i+1 & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}_{3\times 3}$$ is hermitian matrix.

$$Q = \frac{1}{2}(A - A^{\theta}) = \frac{1}{2}\begin{bmatrix} 2i & 2+2i & 6-4i \\ 2i-2 & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix} i & 1+i & 3-2i \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$ is skaw-hermitian matrix

* Inverse of Unitary matrix is unitary.

Proof :- Let 'A' be a unitary matrix.

$$A A^{\theta} = A^{\theta} A = I$$

Consider :- $(A A^{\theta})^{-1} = (I)^{-1}$

$$(A^{\theta})^{-1}(A^{-1}) = I$$

$$(A^{-1})^{\theta}(A^{-1}) = I$$

$\therefore A^{-1}$ is unitary.

* Transpose of a unitary matrix is unitary.

* Product of two unitary matrix is unitary.

* If $A = \begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}_{3\times 3}$ with $X = \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}_{3\times 1}$

find hermitian matrix of H.

Sol:-

$$A X = \begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0+i+0 \\ -i^2+1+2i^2 \\ 0+2i-2i \end{bmatrix}_{3\times 1} = \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

Hermitian form of $H = (\bar{X})^T A X$

$$\bar{X} = \begin{bmatrix} -i \\ 1 \\ i \end{bmatrix}_{3\times 1} \Rightarrow (\bar{X})^T = \begin{bmatrix} -i & 1 & i \end{bmatrix}_{1\times 3}$$

$$(\bar{x})^T A X = \begin{bmatrix} -i & 1 & i \end{bmatrix}_{1\times 3} \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}_{3\times 1} = \begin{bmatrix} -i^2 + 0 + 0 \end{bmatrix}$$

$$(\bar{x})^T A X = [1].$$

★. Elementary Transformations:-

→ Interchanging the two rows/column.

→ K(R) or K(Column) where K is non-zero scalar

→ $KR_1 + R_2$ / $KC_1 + C_2$

Rank of matrix :-

Let 'A' be an $m \times n$ matrix.

→ If A is null matrix then Rank of $A = \rho(A) = 0$

→ If A is non-zero matrix

Let 'r' be rank of A.

Then:-

★ There exist atleast one matrix in A of order 'r' which is not zero.

★. Every minor in A of order greater than 'r' is zero.

It is written rank of A = r.

Q. Find the rank of $A \cdot \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

Sol:-

$$|A| = -1(18-1) - 0(9+5) + 6(3+30)$$

$$\cancel{|A| = -17 + 99 = 82}, \quad |A| = -17 + 6(33)$$

$$|A| = -17 + 198 = 181$$

$\therefore$ Rank of A is 3.

Q.) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$$|A| = 1(24-25) - 2(18-20) + 3(15-16)$$

$$= 1(-1) - 2(-2) + 3(-1) = -1 + 4 - 3 = 0$$

Rank of $A < 3$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

$\therefore$ Rank of $A = 2$.

Q.) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}_{3\times 4}$

matrix order 3×4

Its rank $\leq \min(3,4) = 3$

Highest order minor will be 3.

Condider minor $\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix} = B$

$\det B \Rightarrow 1(0-49) - 2(0-56) + 3(35-48)$

$\det B = -49 + 112 + 3(-13)$

$= -49 + 112 - 39 = -88 + 112$

$= 24 \neq 0$

$\therefore$ Rank of $A = 3$

Important points:

→ Every matrix will have rank.

→ Rank of a matrix is unique.

→ A is non-zero matrix then, Rank of $A \geq 1$

→ If A is matrix of order $m \times n$ then Rank$(A) \leq \min(m,n)$

→ I_n ; Rank of $I_n = n$

→ If A is matrix of order n & A is non-singular ($\det A \neq 0$) then Rank$(A) = n$.

14/12/2020

***. Echelon form of a matrix:**

A matrix is said to be Echelon form.

Properties:

) Zero rows, if any zero row are below any non-zero row

$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(2) The first non-zero ~~row~~ row entry in each non-zero row is '1'. (optional)

(3) The no, of zero rows below the first non-zero element in a row is less than the no, of rows zeros is the next row.

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2^{nd} row zeros $<$ 3^{rd} row zeros.

Rank of Matrix:

The no, of non-zero rows in the row echelon form of A is the rank of A is (A).

Ex:- $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times4} \Rightarrow \boxed{\rho(A) = 2}$

$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4\times5} \Rightarrow \boxed{\rho(A) = 2}$

Q.) Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4}$ into Echelon form and find its rank.

Sol:- $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}_{4\times4}$
$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This in the Echelon form.

No, of non-zero row = 3

$\therefore$ Rank of A = 3.

Q.) $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}_{3\times 4}$ using echelon form and find its rank.

Sol:- $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}_{3\times 4}$

$R_1 \leftrightarrow R_3$

$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \end{bmatrix}_{3\times 4}$

$R_3 \to R_3 - 5R_1$

$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \end{bmatrix}$ $R_3 \to R_3 - 8R_2$

$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -12 & -4 \end{bmatrix}$

This is in Echelon form.

No, of non-zero row = 3

$\therefore$ Rank of A = 3.

) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 18 \end{bmatrix}$ if rank = 2 then find K.

Sol: Rank of $A < 3$ then $|A| = 0$.

$|A| = 1(18k - 42) - 2(36 - 21) + 3(12 - 3k)$

$\Rightarrow 18k - 42 - 2(15) + 36 - 9k = 0$

$\Rightarrow 9k - 42 - 30 + 36 = 0$

$\Rightarrow 9k = 36$

$\boxed{k = 4}$

Q.) Find k such that rank of A = 2

Sol. $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 3R_1$

$\Rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\Rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & 2-k & 0 \end{bmatrix}$ Since Rank of A = 2 then no. of non-zero rows = 2 then 3rd row is zero row.

then $2 - k = 0$.

$\boxed{k = 2}$

*. Reduction to normal form :-

$A_{m\times n} \longrightarrow$ 'r' rank.

↳ reduced to a form I_r (or) $[I_r \ 0]$

(or) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

If $r=2 \Rightarrow A_{m\times n} \rightarrow [I, 0]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$A_{n\times n} \Rightarrow [I, 0]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & \end{bmatrix}_{n\times n}$$

normal form (or) first canonical form of a matrix.

Q.) Reduce a matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}_{3\times 4}$ to canonical form (normal form) and its rank.

Sol:-

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}_{3\times 4}$$

$C_1 \leftrightarrow C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}_{3\times 4}$$

$R_3 \rightarrow R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix} \quad R_2 \rightarrow R_2/2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_3 \to C_3 - 2C_1$$

$$C_4 \to C_4 + 2C_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_2 \to C_2/2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_3 \to C_3 - C_2$$

$$C_4 \to C_4 - 3C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 2$.

Q.) $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}_{4\times 4}$

$R_3 \to R_3 - R_1$; $R_4 \to R_4 - R_1$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_3$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad C_1 \to C_1/2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 - 3C_1 \\ C_4 \to C_4 - 4C_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3/4$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \cancel{C_3 \leftrightarrow} \; R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \leftrightarrow C_4 - C_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$$

This in the normal form.

Rank of $A = 3$.

13/11/2020

* Normal form of the type P and Q.

If $A_{m\times n}$ of rank is 'r' $\boxed{\rho(A) = r}$

The two non singular matrix of P & Q.

$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ then it is called PAQ form of matrix A.

<u>Working Procedure</u>:-

(1) Write $A_{m\times n} = \overset{\nearrow \text{Prefactor}}{I_m} A_{m\times n} \overset{\nearrow \text{post factor}}{I_n}$

(2) Reduce the matrix A on L.H.S to normal form by applying elementary transporation.

(3) Every Elementary row operation on A matrix accompisment by same operation on the pre factor of R.H.S.

(4) Column operation post factor of R.H.S

Note: $I = PAQ$

$P^{-1}I = PP^{-1}AQ$

$P^{-1} = AQ$

$P^{-1}Q^{-1} = AQQ^{-1}$

$P^{-1}Q^{-1} = A \Rightarrow \boxed{A^{-1} = PQ}$

Example:

→ Obtain non-singular matrix P & Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ and obtain its rank.

Sol:- $A_{m\times n} = A_{3\times 3} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

$$A_{3\times 3} = I_{3\times 3} A_{3\times 3} I_{3\times 3}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Pre factor}} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{post factor.}}$$

Row operation:

$R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of A = 2.

Column Operations:-

$C_2 \to C_2 - C_1$; $C_3 \to C_3 - 2C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$C_3 \to C_3 - C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \end{bmatrix} = PAQ.$$

$\therefore$ Rank of $A = 2$.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} ; \ |P| \neq 0$$

$$Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} ; \ |Q| \neq 0$$

$\therefore$ P & Q are non-singular matrix.

(vi) $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4}$

Sol: $A_{3\times 4} = I_{3\times 3} A \, I_{4\times 4}$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

Row Operations:

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \, I_4$$

Column operation:

$C_2 \to C_2 - 3C_1; \quad C_3 \to C_3 - 6C_1; \quad C_4 \to C_4 + C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + C_2$; $C_4 \to C_4 - 2C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_2 \ \ O] = PAQ$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} ; \quad Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$|P| = 1 \neq 0$; $|Q| \neq 0$

$\therefore$ P & Q are non-singular matrix.

Rank of A = 2.

★. Inverse of A matrix by elementary transformation

Inverse by Gauss-Jordan method

Suppose A is non-singular matrix of order, where

$$\boxed{A = I_n A}$$

Apply the elementary transformations (Row operation) L.H.S A and R.H.S I_n

$$A = I_n A \xrightarrow{\text{row operation}} \boxed{I_n = BA}$$

B is inverse of A.

Q) Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}_{3\times 3}$

Sol:- $A_{3\times3} = I_{3\times3} A_{3\times3}$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - 3R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$I_n = BA$

$\therefore$ B is inverse of A.

16/12/2020

System of linear $Eq^{n's}$:

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b \quad \text{—(1)}$$

Consider the system of m linear equations is 'n' Unknowns

$x_1, x_2, \dots x_n$.

(or)

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}\right\} (2)$$

$a_{ij}, b_1, \dots b_n \Rightarrow$ Constants.

$\underbrace{x_1, x_2, \dots x_n \rightarrow n}_{\text{Solutions}}$

(1.) Is the solution unique.

(2.) If how many solution are there.

(3.) In the most general solution how many independent parameters.

From ② Eqⁿ:-

→ If there is atleast one solution. The system is consistent.

- The system of Eqⁿ(2) can be writ into the matrix form: $AX = B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

- The matrix $[AB]$ is called the augmented matrix of the system ②.

By reducing $[AB]$ to its row echelon form.

- If $B = 0$ is ③ the system is homongenous.

$B \neq 0 \Rightarrow$ non-homongenous.

- $AX = 0$ is always consistent.

Since $X = 0$.

$x_1 = x_2 \cdots = x_n = 0$ is always a solution.

↙

Trivial solution

Given:-

$AX = 0$ but $X \neq 0$ such solution is called non-trivial solution.

→ For Non-homogenous system.

- the system $AX = B$ i.e, $B \neq 0$ is Consistent that is it has a solution unique or infinite solution.

$\Leftrightarrow$ Rank of A = Rank of $[AB]$

(1.) If Rank of A = Rank of $[AB]$ $= r <$ no. of unknowns (n)

the system is consistent but there exist infinite solutions.

(2.) If $P(A) = P([AB]) = r = n$

system is consistent and unique solution.

(3.) If $P(A) \neq P([AB])$ then system is inconsistent such that no solution.

Example:- Find whether the following system of eq^n's are Consistent or not if so solve them.

$$x + y + z = 4$$

$$2x + 5y - 2z = 3$$

$$x + 7y - 7z = 5$$

Sol:- $AX = B$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

$P([AB]) = 3$; $P(A) = 2$

$P([AB]) \neq P(A)$

$\therefore$ The system is inconsistent.

$\therefore$ Hence no solution.

Q.) $x+y+z=6$

$2x+3y-2z=2$

$5x+y+2z=13$

Sol:- $AX = B$

$[AB]$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & -2 & 2 \\ 5 & 1 & 2 & 13 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -4 & -3 & -17 \end{bmatrix} \quad R_3 \to R_3 + 4R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -19 & -57 \end{bmatrix}$$

$$\rho([AB]) = 3 = \rho(A)$$

System is consistent and unique solution.

$\boxed{\cancel{x = y = z}}$ $\Rightarrow$ $-19z = -57$

$$\boxed{z = 3}$$

$y - 4z = -10$

$y = -10 + 12$

$\boxed{y = 2}$

$x + y + z = 6$

$x = 6 - 5$

$\boxed{x = 1}$

17|12|2020

Q.) $5x + 3y + 7z = 4$ — (1)

$3x + 26y + 2z = 9$ — (2)

$7x + 2y + 10z = 5$ — (3)

Sol:- $AX = B$

$$[AB] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_2 \rightarrow 5R_2 - 3R_1$

$R_3 \rightarrow 5R_3 - 7R_1$

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_3 \rightarrow 11R_3 + R_2$

15	130	10	45
15	9	21	12
0	121	-11	33
35	10	50	25
35	21	49	28
0	-11		

$$\sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow (1)$$

$$P([AB]) = 2 = P(A)$$

The system is consistent and has infinite solutions.

From (1) :- $5x + 3y + 7z = 4$

$\cancel{3x+}$ $121y - 11z = 33$

$11y - z = 3$

Let :- $\boxed{z = k}$ $\boxed{y = \frac{3+k}{11}}$

$$5x + 3\left(\frac{3+k}{11}\right) = 4 - 7k$$

$$5x = 4 - 7k - \left(\frac{9+3k}{11}\right) = \frac{44 - 77k - 9 - 3k}{11}$$

$$x = \frac{35 - 80k}{55} = \frac{7 - 16k}{11}$$

$$\boxed{x = \frac{7-16k}{11}}$$

Q.) For what values λ, μ the eq^n's are simt.

$$\left.\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}\right\}$$

(i.) no solution

(ii.) unique

(iii.) infinite solution.

Sol:- $AX = B$

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

(i.) no solution :- condition: $P([AB]) \neq P(A)$

if $\lambda = 3$; $\mu \neq 10$

then $P(A) = 2$; $P([AB]) = 3$.

Then system is inconsistent ; no solutions for $\lambda = 3$; $\mu \neq 10$.

(ii.) Condition :- $P(A) = P([AB]) = 3$

then $\lambda \neq 3$; $\mu \neq 10$

The system is consistent and unique solution

iii) Condition :- $P(A) = P([AB]) < 3$

Then $\lambda = 3$; $\mu = 10$.

then $P(A) = P([AB]) = 2 < 3$

then the system is consistent, infinite solutions.

1. $AX = B$ if $B = 0$; $AX = 0 \Rightarrow$ trivial solutions.

$X \neq 0$ non-trivial solutions.

(1.) $AX = 0$

Let $P(A) = r$; $P([AB]) = r_1$

Since $B = 0 \Rightarrow r = r_1 \Rightarrow r = n$

only trivial solutions. $x_1 = x_2 = x_3 \dots = x_n = 0$

(2.) $r < n$ then non-trivial solution.

The system has an infinite solution no, of non-trivial solutions.

(3.) The no, of eq^n is less than the no, of unknows then it has non-trivial solutions.

(4.) No, of eqn's = No, of variables for the solutions then the trivial solutions is the determant of coefficient is zero.

A. Linear dependence and independence of vectors:

A set $\{x_1, x_2, \dots x_r\}$ of r vectors is said to be linearly dependent set if there exist 'r' scalar $k_1, k_2, \dots k_r$ not all zero such that

$$k_1 x_1 + k_2 x_2 + \dots + k_r x_r = 0$$

$$k_1 = k_2 \dots k_n = 0 \Rightarrow \{x_1, x_2, \dots x_n\} \text{ are L.I}$$

Q:) Solve the system of eq^n's:-

$$x + y - 3z + 2w = 0$$

$$2x - y + 2z - 3w = 0$$

$$3x - 2y + z - 4w = 0$$

$$-4x + y - 3z + w = 0$$

Sol:- $AX = 0$; $B = 0$

$$A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & -1 & 2 & -3 \\ 3 & -2 & 1 & -4 \\ -4 & +1 & -3 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$; $R_4 \to R_4 + 4R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & +8 & -7 \\ 0 & -5 & 10 & -10 \\ 0 & 5 & -15 & 9 \end{bmatrix} \quad R_3 \to R_3/-5$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & +8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 5 & -15 & 9 \end{bmatrix}$$

$$R_4 \to R_4 - 5R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & +8 & -7 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -5 & -1 \end{bmatrix}$$

$$R_3 \to 3R_3 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -4 & -7 \\ 0 & 0 & +2 & -1 \\ 0 & 0 & -5 & -1 \end{bmatrix}$$

$$R_4 \to 2R_4 + 5R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & -3 & -4 & -7 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

This in echelon form. no, of non-zero rows = 4

No, of variables = 4

$\boxed{r = n}$ It has trivial solutions.

$\therefore x = y = z = w = 0$.

Q.) Solve the eq^{n}s

$x+y-2z+3w=0$

$x-2y+z-w=0$

$4x+y-5z+8w=0$

$5x-7y+2z-w=0$

Sol:- $AX=B \Rightarrow B=0$

$$A=\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & -2 & 1 & -1 \\ 4 & 1 & -5 & 8 \\ 5 & -7 & 2 & -1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 4R_1$; $R_4 \to R_4 - 5R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -12 & 12 & -16 \end{bmatrix} \quad R_4 \to R_4/4$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \\ 0 & -3 & 3 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$; $R_4 \to R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$; $n=4$

$r < n$

The system has infinite no. of non-trivial solutions

No. of independent solution $= n - r = 2$

$x + y - 2z + 3w = 0 \quad \text{---} (1)$

$-3y + 3z - 4w = 0 \quad \text{---} (2)$

Let $z = k_1$; $w = k_2$

From (2) :- $-3y = 4k_2 - 3k_1$

$$\boxed{y = \frac{3k_1 - 4k_2}{3}}$$

$\Rightarrow x = -3k_2 + 2k_1 - \left(\frac{3k_1 - 4k_2}{3}\right)$

$$x = \frac{-9k_2 + 6k_1 - 3k_1 + 4k_2}{3} = \frac{3k_1 - 5k_2}{3}$$

$$\boxed{x = \frac{3k_1 - 5k_2}{3}}$$

23/12/2020

★. Solution of linear eq^n system - Direct method.

Direct method	Iteration methods.
(1.) Gauss eliminations	(1.) Jacobi's iteration method.
	(2.) Gauss - seidal method.

Gauss elimination:-

n - linear eq^n's & n - variables.

Ex:- $n = 3$

Consider system:-

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$[AB] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{21} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$$

$$R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a^*_{21} & a^*_{23} & b^*_2 \\ 0 & a^*_{32} & a^*_{33} & b^*_3 \end{bmatrix}$$

$\boxed{a_{11} \neq 0} \to a_{11}$ is first pivot element

$$R_3 \to R_3 - \frac{a^*_{32}}{a^*_{21}} R_2$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a^*_{22} & a^*_{23} & b^*_2 \\ 0 & 0 & a^{**}_{33} & b^{**}_3 \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a^*_{22}x_2 + a^*_{23}x_3 = b_2$$

$$a^{**}_{33}x_3 = b^{**}_3$$

$$\boxed{x_3 = \frac{b^{**}_3}{a^{**}_{33}}}$$

24/12/2020

* Solve system of Eq^n's

$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$

Sol:-

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 - 3R_1 \\ R_3 \to 2R_3 - R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{bmatrix} \quad R_3 \to R_3 - 7R_2$$

$$\begin{array}{cccc} 6 & 4 & 6 & 36 \\ 6 & 3 & 3 & 30 \\ \hline 0 & 1 & 3 & 6 \end{array}$$

$$\begin{array}{cccc} 2 & 8 & 18 & 32 \\ 2 & 1 & 1 & 10 \\ \hline 0 & 7 & 17 & 22 \end{array}$$

$$\begin{array}{cccc} 0 & 7 & 17 & 22 \\ 0 & 7 & 21 & 42 \\ \hline 0 & 0 & -4 & -20 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{bmatrix} \quad R_3 \to R_3/-4$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\boxed{x_3 = 5}$$

$$x_2 + 3x_3 = 6$$
$$x_2 = 6 - 15 = -9$$

$$\boxed{x_2 = -9}$$

$$2x_1 + x_2 + x_3 = 10$$
$$2x_1 = 14 \qquad \boxed{x_1 = 7}$$

Q) $x+y+z=6$

$3x+3y+4z=20$

$2x+y+3z=13$

Sol:- $[AB] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$; $R_3 \to R_3 - 2R_1$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{bmatrix}$ $R_3 \leftrightarrow R_2$

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\boxed{z=2}$

$-y+z=1$ $\boxed{x=3}$

$\boxed{y=1}$

** Iterative method:

Gauss-seidel method.

Ex:- $n=3$

$$\textcircled{1}\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Diagonally dominant condition:-

$|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$

Ex:- $10x + y - z = 5$
$8x - 10y + z = 8$ } system.
$x - y + 10z = 10$

Diagonally dominant Condition:-

$|10| > |1| + |-1|$

$|-10| > |8| + |1|$

$|10| > |1| + |-1|$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$ —— (A)

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ —— (B)

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$ —— (C)

(A) $\Rightarrow x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right]$ —— (a)

B $\Rightarrow x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right]$ —— (b)

C $\Rightarrow x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right]$ —— (c)

Let the initial approx solutions

be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$ $(x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0)$

in (a)

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

$$\boxed{x_1^{(1)} = \frac{b_1}{a_{11}}}$$

This is taken as first approx of x_1.

Substituting $x_1^{(1)}$ for x_1, $x_3^{(0)}$ for x_3

in (b).

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right) - a_{23}(0)\right]$$

$$\boxed{x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right]}$$

This is taken first approx of x_2

Substituting $x_1^{(1)}$ for x_1; x_2 for $x_2^{(1)}$.

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}\left(\frac{b_1}{a_{11}}\right) - a_{32}\left(\frac{1}{a_{22}}\left(b_2 - a_{21}\left(\frac{b_1}{a_{11}}\right)\right)\right)\right.$$

Proceeding into very we set success.

The $(k+1)^{th}$ iterative very we are given by :-

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right]$$

Q) $10x+y+z=12$

$2x+10y+z=13$

$2x+2y+10z=14$

Sol:- Diagonally dominant condition :-

$|10| > |1|+|1|$

$|10| > |2|+|1|$

$|10| > |2|+|2|$

Then:- $x = \frac{1}{10}[12-y-z]$ —(1)

$y = \frac{1}{10}[13-2x-z]$ —(2)

$z = \frac{1}{10}[14-2x-2y]$ —(3)

We start iteration by taking $y=0$, $z=0$ in(1) to get.

$$\boxed{x^{(1)} = 1.2}$$

Putting $x = x^{(1)} = 1.2$, $z = 0$ in (2) we get

$$y^{(1)} = \frac{1}{10}[13 - 2.4] = 1.06$$

Putting $x = x^{(1)} = 1.2$, $y = y^{(1)} = 1.06$ in (3) we get

$$z^{(1)} = \frac{1}{10}[14 - 2(1.2) - 2(1.06)]$$

$$z^{(1)} = \frac{1}{10}[14 - 2.4 - 2.12] = 0.95$$

Now taking $y^{(1)}$, $z^{(1)}$ as the initial values in (1) we get:-

$$x^{(2)} = \frac{1}{10}[12 - 1.06 - 0.95]$$

$$\boxed{x^{(2)} = 0.999}$$

Taking $x = x^{(2)}$ and $y = y^{(2)}$; $z = z^{(1)}$ we get in eq(2)

$$y^{(2)} = \frac{1}{10}[13 - 2x^{(2)} - z^{(1)}]$$

$$y^{(2)} = \frac{1}{10}[13 - 2(0.999) - 0.95]$$

$$\boxed{y^{(2)} = 1.005}$$

Next, taking $x = x^{(2)}$; $y = y^{(2)}$ in (3)

$$z^{(2)} = \frac{1}{10}[14 - 2x^{(2)} - 2y^{(2)}]$$

$$z^{(2)} = \frac{1}{10}[14 - 2(0.999) - 2(1.005)]$$

$$\boxed{z^{(2)} = 0.999}$$

Again taking $x^{(2)}, y^{(2)}, z^{(2)}$ as the initial values, we get

$$x^{(3)} = \frac{1}{10}(12 - 1.005 - 0.999) = 0.9996 = 1.00$$

$$y^{(3)} = \frac{1}{10}(13 - 2.0 - 0.999) = 1.0001 = 1.00$$

$$z^{(3)} = \frac{1}{10}(14 - 2.0 - 2.0) = 1.00$$

Similarly, we find the fourth approximations of x, y, z and get them as $x^{(4)} = 1.00$, $y^{(4)} = 1.00$, $z^{(4)} = 1.00$

We tabulate the results as follows.

Variable	1st approx	2nd approx	3rd approx	4th approx
x	1.20	0.999	1.00	1.00
y	1.05	1.005	1.00	1.00
z	0.95	0.999	1.00	1.00

Thus the solution of the given system of equations is $x = 1, y = 1; z = 1$.

Example:-2:- Solve the system of equations using Gauss-Seidel iteration method $27x + 6y - z = 85; 6x + 15y + 2x = 72;$ $x + y + 54z = 110$.

Sol:- The given system is diagonally dominant, and we rewrite it as:

$$x = \frac{1}{27}(85 - 6y + z) \quad \text{—(1)}$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad \text{—(2)}; \quad z = \frac{1}{54}(110 - x - y) \quad \text{—(3)}$$

We start the iteration by taking $y=0$ and $z=0$ in (1),

$x^{(1)} = 3.15$

Putting $x^{(1)} = 3.15$, $z=0$ in (2) we get.

$y^{(1)} = 3.54$

Next putting $x^{(1)} = 3.15$ and $y^{(1)} = 3.54$ in (3) we get.

$z^{(1)} = 1.91.$

Now, taking $y^{(1)} = 3.54$ and $z^{(1)} = 1.91$ in (1) we get $x^{(2)} = 2.43$

Taking $x = x^{(2)}$; $z = z^{(1)}$ in (2) we get $y^{(2)} = 3.57.$

Taking $x = x^{(2)} = 2.43$ and $y^{(2)} = y = 3.57$ in (3)

we get $z^{(2)} = 1.926.$

Similarly, we obtain:

$x^{(3)} = 2.426$, $y^{(3)} = 3.572$, $z^{(3)} = 1.926$

$x^{(4)} = 2.425$, $y^{(4)} = 3.573$, $z^{(4)} = 1.926$

and $x^{(5)} = 2.425$, $y^{(5)} = 3.573$, $z^{(5)} = 1.926$

$\therefore$ The solution of the given system of equations is

$x = 2.4$; $y = 3.6$; $z = 2$.